DIFFERENTIAL TOPOLOGY OF MANIFOLDS ADMITTING ROUND FOLD MAPS

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Abstract. In this paper, under additional conditions, we study the homeomorphism and diffeomorphism types of manifolds admitting *round fold maps*, or *stable fold maps* with concentric singular value sets introdued by the author [10].

1. Introduction

Fold maps are important in generalizing the theory of Morse functions. Studies of such maps were started by Whitney ([23]) and Thom ([22]) in the 1950's. A *fold map* is a C^∞ map whose singular points are of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} {x_k}^2 - \sum_{k=m-i+1}^{m} {x_k}^2)$$

for two positive integers $m \geq n$ and an integer $0 \leq i \leq m-n+1$. A Morse function is a fold map of course. For a fold map from a closed C^∞ manifold of dimension m into a C^∞ manifold of dimension n (without boundary) the followings hold where $m \geq n \geq 1$.

(1) The *singular set*, or the set of all the singular points, is a closed C^∞ submanifold of dimension $(n-1)$ of the source manifold.
(2) The restriction map to the singular set is a C^∞ immersion of codimension 1.

We also note that if the restriction map to the singular set is a immersion with normal crossings, then it is *stable* (stable maps are important in the theory of global singularity; see [6] for example). In [10], such a fold map is defined as a *stable* fold map.

Since around the 1990's, fold maps with additional conditions have been actively studied. For example, in [2], [5], [17], [18] and [20], *special generic* maps, which are fold maps whose singular points are of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m} {x_k}^2)$$

for two positive integers $m \geq n$, were studied. In [20], Sakuma studied *simple* fold maps, which are fold maps such that fibers of singular values do not have any connected component with more than one singular points (see also [16]). For example, special generic maps are simple. In [12], Kobayashi and Saeki investigated

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topology of *stable* maps including fold maps which are stable into the plane. In [19], Saeki and Suzuoka found good properties of manifolds admitting stable maps whose regular fibers, or fibers of regular values, are disjoint unions of spheres.
Later, in [10], *round* fold maps, which will be mainly studied in this paper, were introduced. A *round* fold map is a fold map satisfying the followings.

(1) The singular set is a disjoint union of standard spheres.
(2) The restriction map to the singular set is a C^∞ embedding.
(3) The singular value set is a disjoint union of spheres embedded concentrically.

Fold maps satisfying the definition of a round fold map have appeared in previous studies as the followings.

- A lot of special generic maps on homotopy spheres. (See also [17])
- Fold maps represented as FIGURE 7 of [12] and Figure 8 of [19].

(Algebraic) topological properties of round fold maps were studied in Theorem 1, Theorem 2 (homology groups) and Theorem 3 (homotopy groups) of [10] under appropriate conditions.
Now, how about the homeomorphism or diffeomorphism types of manifolds admitting round fold maps? In this paper, we study such problems under appropriate conditions.
This paper is organized as the following.
Section 2 is for preliminaries. We recall *fold maps* and introduce *stable* fold maps. We also recall *special generic* maps and *simple* fold maps. We also review the *Reeb space* of a smooth map, which is the space consisting of all the connected components of the fibers of the map. Last we introduce terminologies on simple fold maps and a result of [11].
In section 3, we review [10]. We recall *round* fold maps and some terminologies on round fold maps such as *axes*, which are rays originating from points in the connected components of the regular value sets located in the center of the target Euclidean spaces and *proper cores*, which are closed discs embedded in the previous connected components of the regular value sets (FIGURE 1). We also recall a result on round fold maps whose regular fibers consist of disjoint unions of spheres. We review Theorem 3 of [10], which state that some homotopy groups of manifolds admitting such maps are determined by the topological properties of the Reeb spaces (Proposition 5). We also introduce Theorem 2 of [11], which is a result for round fold maps having regular fibers PL homeomorphic to products of two spheres (Proposition 6).
In section 4, we introduce round fold maps on spheres and construct round fold maps on closed C^∞ manifolds having the structures of bundles over S^n where $n \geq 2$ (Theorem 1). We also study the homeomorphism or diffeomorphism types of manifolds admitting round fold maps into the plane with the inverse images of axes C^∞ diffeomorphic to cylinders (Theorem 3, Theorem 4 and Theorem 5).
In section 5, we study the diffeomorphism types of manifolds of dimensions m admitting round fold maps into $\mathbb{R}^n$ under additional conditions where $n \geq 2$ and $m \geq 2n$ hold. In subsection 5.1, under appropriate conditions, we construct a new round fold map from two round fold maps on closed and connected oriented manifolds on the connected sum of the two manifolds (Proposition 8, Theorem 6 and Theorem 7). Conversely, in subsection 5.2, we decompose a round fold map on a closed and connected oriented manifold into two round fold maps so that the connected sum of the resulting source manifolds is the original source manifold

(Proposition 10 and Theorem 8). In these subsections, we apply generalizations of surgery operations on stable maps from closed and simply-connected C^∞ manifolds into the plane which do not change the diffeomorophism types of the source manifolds in [12] (*R-operations*) and their inverse operations. Last, in subsection 5.3, by applying the obtained results and their proofs, we determine the diffeomorphism types of manifolds admitting round fold maps under differential topological restrictions on fibers and the naturally defined bundle structures of the inverse images of small C^∞ closed tubular neighborhoods of connected components of singular value sets (Theorem 9, Theorem 10 and Theorem 11).

Throughout this paper, we assume that M is a closed C^∞ manifold of dimension m, that N is a C^∞ manifold of dimension n without boundary, that $f: M \rightarrow N$ is a C^∞ map and that $m \geq n \geq 1$. We denote the *singular set* of f, or the set consisting of all the singular points of f, by $S(f)$.

2. Preliminaries

2.1. **Fold maps.** First, we recall *fold maps*, which are simplest generalizations of Morse functions. See also [6], [14] and [15] for example.

Definition 1. For a C^∞ map $f: M \rightarrow N$, $p \in M$ is said to be a *fold* point of f if at p f has the normal form

$$f(x_1, \cdots, x_m) := (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} {x_k}^2 - \sum_{k=m-i+1}^{m} {x_k}^2)$$

and if all the singular points of f are fold, then we call f a *fold map*.

If $p \in M$ is a fold point of f, then we can define $j := \min\{i, m-n+1-i\}$ uniquely in the previous definition. We call p a *fold point of index j* of f. We call a fold point of index 0 a *definite* fold point of f and we call f a *special generic* map if all the singular points are definite fold. For special generic maps, see [2], [5], [17] and [20] for example. Let f be a fold map. Then the singular set $S(f)$ and the set of all the fold points whose indices are i (we denote the set by $F_i(f)$) are C^∞ $(n-1)$-submanifolds of M. The restriction map $f|_{S(f)}$ is a C^∞ immersion.

A Morse function on a closed manifold is a fold map. A Morse function on a closed manifold which has just two singular points is a special generic map. Conversely, a Morse function which is a special generic map on a closed and connected manifold whose dimension is larger than 1 has just two singular points.

In this paper, we sometimes need Morse functions on compact C^∞ manifolds possibly with boundaries. We call a Morse function on such a manifold *good* if it is constant and minimal on the boundary, singular points of it are not on the boundary and at any two distinct singular points, the values are distinct.

We introduce *stable* fold maps.

Definition 2. A fold map $f: M \rightarrow N$ is said to be a *stable* fold map if the restriction $f|_{S(f)}$ is a C^∞ immersion with normal crossings.

Note that a stable fold map on a closed C^∞ manifold is also a *stable* map (for stable maps, see [6] for example).

Note also that a Morse function on a closed C^∞ manifold is a stable fold map if and only if it is good.

We also introduce *simple* fibers of fold maps and *simple* fold maps (see also [16] and [20] for example).

Definition 3. For a fold map f and $p \in f(S(f))$, $f^{-1}(p)$ is said to be *simple* if each connected component of $f^{-1}(p)$ includes at most one singular point of f. f is said to be a *simple* fold map if for each $p \in f(S(f))$, $f^{-1}(p)$ is simple.

Example 1.
(1) A Morse function on a closed manifold is simple if it is good.
(2) A fold map $f : M \to \mathbb{R}^n$ is simple if $f|_{S(f)}$ is a C^{∞} embedding.
(3) Special generic maps are simple.

2.2. **Reeb spaces.** We review the *Reeb space* of a map.

Definition 4. Let X, Y be topological spaces. For $p_1, p_2 \in X$ and for a map $c : X \to Y$, we define as $p_1 {\sim_c} p_2$ if and only if p_1 and p_2 are in the same connected component of $c^{-1}(p)$ for some $p \in Y$. $\sim_c$ is an equivalence relation.
We denote the quotient space $X/{\sim_c}$ by W_c. We call W_c the *Reeb space* of c.

We denote the induced quotient map from X into W_c by q_c. We define $\bar{c} : W_c \to Y$ so that $c = \bar{c} \circ q_c$. W_c is often homeomorphic to a polyhedron.
For example, for a good Morse function, the Reeb space is a graph. For a simple fold map, the Reeb space is homeomorphic to a polyhedron which is not so complex (see Proposition 2).
For a special generic map, the Reeb space is homeomorphic to a C^{∞} manifold. See section 2 of [17]. See also [2] and [5] for example.
In [12], it is proven that for a stable fold map (or stable map) $f : M \to \mathbb{R}^2$ ($m \geq 2$), W_f is homeomorphic to a polyhedron. It is known that for a stable map f, the Reeb space W_f is homeomorphic to a polyhedron. See [8] for example.
The following holds since a stable fold map is stable.

Proposition 1. *For a stable fold map f, the Reeb space W_f is homeomorphic to a polyhedron.*

The following Proposition 2 is well-known and we omit the proof. See [12], [16] and [19] for example. In this paper, we often apply the statements of this proposition implicitly.
We also note that in this paper, an *almost-sphere* of dimension k means a C^{∞} homotopy sphere given by glueing two standard closed discs of dimensions k together by a C^{∞} diffeomorphism between the boundaries.
We often use terminologies on (fiber) bundles in this paper (see also [21]). For a topological space X, an *X-bundle* is a bundle whose fiber is X. A bundle whose structure group is G is said to be a *trivial* bundle if it is equivalent to the product bundle as a bundle whose structure group is G. Especially, a trivial bundle whose structure group is a subgroup of the homeomorphism group of the fiber is said to be a *topologically trivial* bundle. In this paper, a C^{∞} (PL) *bundle* means a bundle whose fiber is a C^{∞} (resp. PL) manifold and whose structure group is a subgroup of the C^{∞} diffeomorphism group (resp. PL homeomorphism group) of the fiber. A *linear* bundle is a C^{∞} bundle whose fiber is a standard disc or a standard sphere and whose structure group is a subgroup of an orthogonal group.

Proposition 2. *Let $f : M \to N$ be a special generic map or a simple fold map or a stable fold map. Then W_f has the structure of a polyhedron and the followings hold.*

(1) *$W_f - q_f(S(f))$ is uniquely given the structure of a C^∞ manifold such that $q_f|_{M-S(f)} : M - S(f) \to W_f - q_f(S(f))$ is a C^∞ submersion. Furthermore, for any compact C^∞ submanifold R of dimension n of any connected component of $W_f - q_f(S(f))$, R is a subpolyhedron of W_f and $q_f|_{{q_f}^{-1}(R)} : {q_f}^{-1}(R) \to R$ gives the structure of a C^∞ bundle whose fiber is a connected C^∞ manifold of dimension $m - n$.*

(2) *The restriction of q_f to the set $F_0(f)$ of all the definite fold points, is injective.*

(3) *f is simple if and only if $q_f|_{S(f)} : S(f) \to W_f$ is injective. Special generic maps are simple.*

(4) *If f is simple, then for any connected component C of $S(f)$, $q_f(C)$ has a small regular neighborhood $N(q_f(C))$ in W_f and ${q_f}^{-1}(N(q_f(C)))$ has the structure of a C^∞ bundle over $q_f(C)$.*

(5) *For any connected component C of $F_0(f)$, any small regular neighborhood of $q_f(C)$ has the structure of a trivial* PL *$[0,1]$-bundle over $q_f(C)$ and $q_f(C)$ corresponds to the 0-section. We can take a small regular neighborhood $N(q_f(C))$ of $q_f(C)$ and ${q_f}^{-1}(N(q_f(C)))$ has the structure of a linear D^{m-n+1}-bundle over $q_f(C)$. More precisely, the bundle structure is given as the following; for the connected component C' of $\partial N(q_f(C))$ satisfying $C' \bigcap q_f(F_0(f)) = \emptyset$, the composition of $q_f|_{{q_f}^{-1}(N(q_f(C))} : {q_f}^{-1}(N(q_f(C))) \to N(q_f(C))$ and the projection to $q_f(C)$ (the subbundle corresponding to the 0-section) or C' (the subbundle corresponding to the fiber $\{1\} \subset [0,1]$) .*

(6) *Let f be simple. Let $m - n \geq 1$. If $m - n = 1$, then we also assume that M is orientable.*
Then for any connected component C of the set $F_1(f)$ of all the fold points of indice 1, such that for any connected component R of $W_f - q_f(S(f))$ whose closure $\overline{R}$ includes $q_f(C)$, ${q_f}^{-1}(p)$ is an almost-sphere for $p \in R$, any small regular neighborhood of $q_f(C)$ has the structure of a K-bundle over $q_f(C)$ where $K := \{r\exp(2\pi i\theta) \in \mathbb{C} \mid 0 \leq r \leq 1, \theta = 0, \frac{1}{3}, \frac{2}{3}\}$, where the structure group consists of just two elements fixing the point $1 \in K$ and where $q_f(C)$ corresponds to the 0-section. We can take a small regular neighborhood $N(q_f(C))$ of $q_f(C)$ and ${q_f}^{-1}(N(q_f(C)))$ has the structure of a C^∞ bundle over $q_f(C)$ whose fiber is PL *homeomorphic to S^{m-n+1} with the interior of a union of disjoint three standard closed $(m-n+1)$-discs removed. More precisely, the bundle structure is given by the composition of $q_f|_{{q_f}^{-1}(N(q_f(C)))} : {q_f}^{-1}(N(q_f(C))) \to N(q_f(C))$ and the projection to $q_f(C)$ (the subbundle corresponding to the 0-section) or the projection to the subbundle corresponding to the fiber $\{1\} \subset K$.*

Last we introduce terminologies on simple fold maps and a main result of [11]. Here we define a *branched* point of a polyhedron. For a polyhedron of dimension $k \geq 1$, a *branched* point means a point such that every open neighborhood of the point is not homeomorphic to any open set of $\mathbb{R}^k$ or ${\mathbb{R}^k}_+ := \{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}$. If a polyhedron X of dimension k does not have branched points, then it is a manifold having a triangulation and we can define the *interior* Int$X$ and the *boundary* ∂X.

Definition 5 ([11])**.** Let $f : M \to N$ be a simple fold map and $m - n \geq 2$.

(1) Let C be a connected component of $q_f(S(f)) - q_f(F_0(f))$ consisting of non-branched points. Assume that there exists a small regular neighborhood $N(C)$ of C in W_f having the structure of a trivial PL $[-1,1]$-bundle over $q_f(C)$ ($q_f(C)$ corresponds to the 0-section) and that the composition of $q_f|_{{q_f}^{-1}(N(C))} : {q_f}^{-1}(N(C)) \to N(C)$ and the projection to $q_f(C)$ gives ${q_f}^{-1}(N(C))$ the structure of a C^∞ bundle over C whose fiber is PL homeomorphic to $D^{k+1} \times S^{m-n-k} - \mathrm{Int} D^{m-n+1}$ for an integer $1 \leq k < m-n$. Then C is said to be a *k S-locus*.

(2) Let R be a connected component of $W_f - q_f(S(f))$.
If for any compact subset P in R, $q_f|_{{q_f}^{-1}(P)} : {q_f}^{-1}(P) \to P$ gives the structure of a bundle over P whose fiber is PL homeomorphic to $S^k \times S^{m-n-k}$ for an integer $1 \leq k \leq [\frac{m-n}{2}]$, then R is said to be a *k S-region* or an *$m-n-k$ S-region* (note $k = m-n-k$ if $m-n$ is even and $k = \frac{m-n}{2}$).
If for $p \in R$, ${q_f}^{-1}(p)$ is an almost-sphere, then R is said to be an *AS-region*.

For example, on $S^{k_1} \times S^{k_2}$ ($k_1, k_2 \in \mathbb{N}$, $k_1, k_2 \geq 2$), there exists a good Morse function $f : S^{k_1} \times S^{k_2} \to \mathbb{R}$ with four singular points such that W_f contains two $k_1 - 1$ ($k_2 - 1$) S-loci.
Let $f : M \to N$ be a simple fold map and $p \in S(f)$. Let $q_f(p)$ be in a k S-locus. If $k+1 \leq [\frac{m-n+1}{2}]$, then $p \in F_{k+1}(f)$ and if $k+1 > [\frac{m-n+1}{2}]$, then $p \in F_{m-n-k}(f)$.

Definition 6 ([11])**.** If a simple fold map $f : M \to N$ ($m - n \geq 2$) satisfies the followings, then f is said to be a *normal simple fold map with regular fibers of two spheres*.

(1) Any connected component of $q_f(S(f))$ in $W_f - q_f(F_0(f))$ consisting of non-branched points is an S-locus .
(2) For any connected component R of $W_f - q_f(S(f))$ such that the intersection of the closure $\overline{R}$ and the set of all the branched points of W_f is non-empty, the fiber of each point in R is an almost-sphere.

For example, on $S^{k_1} \times S^{k_2}$ ($k_1, k_2 \in \mathbb{N}$), there exists a good Morse function $f : S^{k_1} \times S^{k_2} \to \mathbb{R}$ with four singular points which is also a normal simple fold map with regular fibers of two spheres.

Definition 7 ([11])**.** Let f be a normal simple fold map with regular fibers of two spheres.

(1) If for any k S-locus C, the connected component R of $W_f - q_f(S(f))$ which is not an AS-region such that C is in the boundary of the closure $\bar{R}$, is a k S-region and the boundary of the closure is a disjoint union of k S-loci, then we say that f *decomposes into S-systems*.

(2) Assume that f decomposes into S-systems.
We say that f *has a family of S-identifications* if there exists a family of small regular neighborhoods $\{N(C_\lambda)\}_{\lambda \in \Lambda}$ of all the S-loci $\{C_\lambda\}_{\lambda \in \Lambda}$ such that the followings hold.
   (a) (The bundle structure of ${q_f}^{-1}(N(C_\lambda))$ over C_λ)
   Let C_λ be a k S-locus. ${q_f}^{-1}(N(C_\lambda))$ has the structure of a C^∞ bundle over C_λ with the fiber PL homeomorphic to $D^{k+1} \times S^{m-n-k} - \mathrm{Int} D^{m-n+1} \subset D^{k+1} \times S^{m-n-k}$ for an integer $1 \leq k < m-n$ such that for the connected component C of $\partial N(C_\lambda)$ in a k S-region, ${q_f}^{-1}(C)$

has the structure of a subbundle of the bundle ${q_f}^{-1}(N(C_\lambda))$ over C_λ with the fiber PL homeomorphic to $\partial D^{k+1} \times S^{m-n-k}$ and that the structure group of ${q_f}^{-1}(C)$ is a subgroup of the C^∞ diffeomorphism group of the fiber, consisting of some C^∞ diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $\partial D^{k+1} \times S^{m-n-k}$ over ∂D^{k+1} inducing PL homeomorphisms on the base space $\partial D^{k+1} = S^k$ $(1 \leq k < m-n)$.

(b) (The bundle structure of ${q_f}^{-1}(R)$ over R for a connected component R of $W_f - \mathrm{Int} \sqcup_{\lambda \in \Lambda} N(C_\lambda)$ in an S-region)
For any connected component R of $W_f - \mathrm{Int} \sqcup_{\lambda \in \Lambda} N(C_\lambda)$ in a k S-region such that the closure of the k S-region is bounded by a disjoint union of k S-loci, ${q_f}^{-1}(R)$ has the structure of a C^∞ bundle over R with the fiber PL homeomorphic to $S^k \times S^{m-n-k}$ such that the structure group is a subgroup of the C^∞ diffeomorphism group of the fiber, consisting of some C^∞ diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $S^k \times S^{m-n-k}$ over S^k inducing PL homeomorphisms on the base space S^k.

(c) (Identifications of pieces of the source manifold on their boundaries)
For any connected component R of $W_f - \mathrm{Int} \sqcup_{\lambda \in \Lambda} N(C_\lambda)$ in a k S-region such that the closure of the k S-region is bounded by a disjoint union of k S-loci, the identification map of the restriction of a bundle ${q_f}^{-1}(R)$ over R satisfying (b) of this definition to any connected component C of the boundary ∂R and the subbundle ${q_f}^{-1}(C)$ of a bundle ${q_f}^{-1}(N(C_\lambda))$ $(C \subset \partial N(C_\lambda))$ over C satisfying (a) of this definition is a bundle map; the structure groups are a subgroup of the C^∞ diffeomorphism group of the fiber, consisting of some C^∞ diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $\partial D^{k+1} \times S^{m-n-k}$ over ∂D^{k+1} inducing PL homeomorphisms on the base space $\partial D^{k+1} = S^k$. (Here the base space of the fiber of the bundle satisfying (a) means ∂D^{k+1} and the base space of the fiber of the bundle satisfying (b) means S^k. We should be careful in the case where $k = m-n-k$.)

For example, on $S^{k_1} \times S^{k_2}$ $(k_1, k_2 \in \mathbb{N})$, there exists a good Morse function $f : S^{k_1} \times S^{k_2} \to \mathbb{R}$ with four singular points which has a family of S-identifications. Now we introduce a main result of [11].

Proposition 3 ([11]). *Let M be a closed C^∞ manifold of dimension m, N be a C^∞ manifold of dimension n without boundary and $f : M \to N$ be a simple fold map. Let $m - n \geq 2$.*
We assume that f has a family of S-identifications.

Then there exist a compact PL *manifold W of dimension $m+1$ such that $\partial W = M$, a polyhedron V and continuous maps $r : W \to V$ and $s : V \to W_f$ and the followings hold.*

(1) *There exist a triangulation of W, a triangulation of V and a triangulation of W_f such that r is a simplicial map and that s is a simplicial map and the followings hold.*
   (a) *For each $p \in V$, $r^{-1}(p)$ collapses to a point and r is a homotopy equivalence.*

(b) *If p is in the closure of an AS-region in W_f, then $s^{-1}(p)$ is a point. If p is in an AS-region in W_f and $q \in s^{-1}(p)$, then $r^{-1}(q)$ is* PL *homeomorphic to D^{m-n+1}.*

(c) *If p is in a k S-region in W_f whose closure is bounded by a disjoint union of k S-loci, then $s^{-1}(p)$ is* PL *homeomorphic to S^k. If p is in a k S-region in W_f whose closure is bounded by a disjoint union of k S-loci and $q \in s^{-1}(p)$, then $r^{-1}(q)$ is* PL *homeomorphic to $D^{m-n-k+1}$.*

(2) *W collapses to a subpolyhedron V' such that $r|_{V'} : V' \to V$ is a* PL *homeomorphism.*

If M is orientable, then we can construct W as an orientable manifold.

Corollary 1 ([11])**.** *In the situation of Proposition 3, let M be connected and $i : M \to W$ be the natural inclusion. Then*

$$r_* \circ i_* : \pi_k(M) \to \pi_k(V)$$

gives an isomorphism for $0 \leq k \leq m - \dim V - 1$.

3. Notes on round fold maps

In this section, we review *round* fold maps. See also [10].
We introduce two definitions of a round fold map and show equivalence of them.
First we recall C^∞ *equivalence*. For two C^∞ maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, we say that they are C^∞ *equivalent* if there exist C^∞ diffeomorphisms $\phi_X : X_1 \to X_2$ and $\phi_Y : Y_1 \to Y_2$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_X} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{\phi_Y} & Y_2 \end{array}$$

For C^∞ equivalence, see also [6] for example.

3.1. **Terms on round fold maps.**

Definition 8 (round fold map)**.** $f : M \to \mathbb{R}^n$ $(m \geq n \geq 2)$ is said to be a *round* fold map if f is C^∞ equivalent to a fold map $f_0 : M_0 \to \mathbb{R}^n$ on a closed C^∞ manifold M_0 such that the followings hold.

(1) The singular set $S(f_0)$ is a disjoint union of standard spheres of dimensions $n-1$ and consists of $l \in \mathbb{N}$ connected components.
(2) The restriction map $f_0|_{S(f_0)}$ is a C^∞ embedding.
(3) Let ${D^n}_r := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^{n} {x_k}^2 \leq r\}$. Then $f_0(S(f_0)) = \sqcup_{k=1}^{l} \partial {D^n}_k$.

We call f_0 a *normal form* of f. We call a ray L from $0 \in \mathbb{R}^n$ an *axis* of f_0 and ${D^n}_{\frac{1}{2}}$ the *proper core* of f_0. Suppose that for a round fold map f, its normal form f_0 and C^∞ diffeomorphisms $\Phi : M \to M_0$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\phi \circ f = f_0 \circ \Phi$. Then for an axis L of f_0, we also call $\phi^{-1}(L)$ an *axis* of f and for the proper core ${D^n}_{\frac{1}{2}}$ of f_0, we also call $\phi^{-1}({D^n}_{\frac{1}{2}})$ a *proper core* of f.

We introduce another definition.

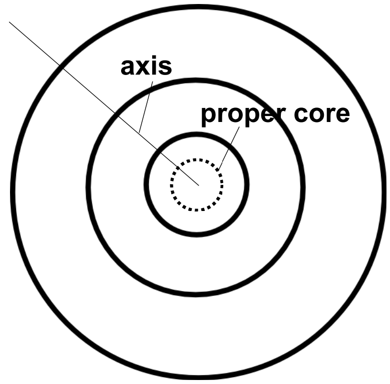

FIGURE 1. An axis and a proper core of a round fold map

Definition 9 (round fold map)**.** Assume that $f : M \to \mathbb{R}^n$ is a fold map and that $m \geq n \geq 2$. f is said to be a *round* fold map if the followings hold.

(1) The singular set $S(f)$ is a disjoint union of standard spheres of dimensions $n-1$.
(2) The restriction map $f|_{S(f)}$ is a C^{∞} embedding.
(3) We denote by $\{U_0\} \sqcup \{U_{\infty}\} \sqcup \{U_{\lambda}\}_{\lambda \in \Lambda}$ (Λ may be empty.) the set of all the connected components of $\mathbb{R}^n - f(S(f))$. The followings hold.
   (a) The closure $\overline{U_0}$ is C^{∞} diffeomorphic to D^n.
   (b) The closure $\overline{U_{\infty}}$ is C^{∞} diffeomorphic to $S^{n-1} \times [0, +\infty)$.
   (c) The closure $\overline{U_{\lambda}}$ is C^{∞} diffeomorphic to $S^{n-1} \times [0, 1]$.

In [10], the following was shown.

Proposition 4. *Two definitions of a round fold map are equivalent.*

Let f be a normal form of a round fold map and $P^{(1)} := {D^n}_{\frac{1}{2}}$. We set $E := f^{-1}(P^{(1)})$ and $E' := M - f^{-1}(\mathrm{Int}P^{(1)})$. We put $F := f^{-1}(p)$ for $p \in \partial P^{(1)}$. We put $P^{(2)} := \mathbb{R}^n - \mathrm{Int}P^{(1)}$, $f_1 := f|_E : E \to P^{(1)}$ and $f_2 := f|_{E'} : E' \to P^{(2)}$. f_1 gives the structure of a trivial C^{∞} bundle over $P^{(1)}$, $f_1|_{\partial E} : \partial E \to \partial P^{(1)}$ gives the structure of a trivial C^{∞} bundle over $\partial P^{(1)}$ and $f_2|_{\partial E'} : \partial E' \to \partial P^{(2)}$ gives the structure of a trivial C^{∞} bundle over $\partial P^{(2)}$. Note that $f(M)$ may not be C^{∞} diffeomorphic to D^n and that $f(M) \bigcap P^{(1)} = \emptyset$ may hold.
We can give E' and $q_f(E')$ the structures of bundles over $\partial P^{(2)}$ as follows.
Since for $\pi_P(x) := \frac{1}{2}\frac{x}{|x|}$ ($x \in P^{(2)}$), $\pi_P \circ f|_{E'}$ is a proper C^{∞} submersion, this map gives E' the structure of a C^{∞} $f^{-1}(L)$-bundle over $\partial P^{(2)}$ (apply Ehresmann's fibration theorem [4]).
$q_f(E')$ has the structure of a bundle over $\partial P^{(2)}$. We consider the following map. Let $p \in q_f(E')$. For $p_1, p_2 \in {q_f}^{-1}(p)$, $\pi_P \circ f(p_1) = \pi_P \circ f(p_2)$. We correspond $\pi_P \circ f(p_1) = \pi_P \circ f(p_2) = \pi_P \circ \bar{f}(p)$ to p. This map from $q_f(E')$ into $\partial P^{(2)}$ gives the structure of a $\bar{f}^{-1}(L)$-bundle since $\pi_P \circ f|_{E'}$ gives the structure of a C^{∞} bundle and $q_f(E')$ is the quotient space of E' by $\sim_f$.
For a round fold map f which is not a normal form, we can define similar maps and consider similar structures of bundles.
We can define the following condition for a round fold map.

Definition 10. Let $f : M \to \mathbb{R}^n$ be a round fold map. Let $\mathcal{C} := \mathrm{PL}, C^\infty$. If the natural projection $\pi_P \circ f|_{E'}$ from the total space of the bundle E' onto the base space $\partial P^{(2)}$ ($\partial P^{(2)}$ is C^∞ diffeomorphic to S^{n-1}) gives the structure of a topologically ($\mathcal{C}$) trivial bundle, then f is said to be *topologically* (*resp.* $\mathcal{C}$) *trivial*. If the natural projection $\pi_P \circ \bar{f}|_{q_f(E')}$ from the total space of the bundle $q_f(E')$ onto the base space $\partial P^{(2)}$ gives the structure of a topologically trivial bundle, then f is said to be *topologically quasi-trivial*.

3.2. **Some facts on round fold maps.** In [10], the following was shown.

Lemma 1 ([10])**.** *Let M be a closed C^∞ manifold of dimension m, N be a C^∞ manifold without boundary of dimension n and $m > n \geq 1$.*
If $m - n = 1$, then we also assume that M is orientable.
Let $f : M \to N$ be a simple fold map. We assume that for each regular value p, $f^{-1}(p)$ is a disjoint union of almost-spheres and that the indices of all the fold points of f are 0 or 1.
Then there exist a compact PL *$(m + 1)$-manifold W such that $\partial W = M$ and a continuous map $r : W \to W_f$ such that $r|_{\partial W}$ coincides with $q_f : M \to W_f$. Furthermore, the followings hold.*

1. *For each $p \in W_f - q_f(S(f))$, $r^{-1}(p)$ is* PL *homeomorphic to D^{m-n+1}.*
2. *There exist a triangulation of W and a triangulation of W_f such that r is a simplicial map.*
3. *For each $p \in W_f$, $r^{-1}(p)$ collapses to a point and r is a homotopy equivalence.*
4. *W collapses to a subpolyhedron ${W_f}'$ such that $r|_{{W_f}'} : {W_f}' \to W_f$ is a* PL *homeomorphism.*

By applying the lemma, the following corollary was shown.

Corollary 2. *In the situation of Lemma 1, let M be connected and $i : M \to W$ be the natural inclusion. Then*

$$q_{f*} = r_* \circ i_* : \pi_k(M) \to \pi_k(W_f)$$

gives an isomorphism for $0 \leq k \leq m - n - 1$.

By applying the corollary, the following proposition was shown (Theorem 3 of [10]).

Proposition 5 ([9], [10])**.** *Let M be a closed and connected C^∞ manifold of dimension m, $f : M \to \mathbb{R}^n$ be a round fold map and $m > n \geq 2$. If $m - n = 1$, then we also assume that M is orientable.*
We assume that $f^{-1}(p)$ is a disjoint union of almost-spheres for each regular value p and that the indices of all the fold points of f are 0 or 1.
Let L be an axis of f and $f_L := f|_{f^{-1}(L)}$. We denote by l_1 the number of loops of the Reeb space W_{f_L} of f_L (in other words, let $H_1(W_{f_L}) \cong \mathbb{Z}^{l_1}$). We denote by l_2 the number of connected components of the fiber of a point in a proper core of f.
Then there exist W and a homotopy equivalence $r : W \to W_f$ as in Lemma 1. Furthermore, $q_f = r \circ i$ gives an isomorphism of homotopy groups $\pi_k(M) \cong \pi_k(W_f)$ for $0 \leq k \leq m-n-1$ and we have the following list where we denote the free group of rank r by F_r.

(1) *When $n \geq 3$, $m \geq 2n$, f is topologically quasi-trivial and we have the followings.*

$$\pi_k(M) \cong \pi_k(W_f) \cong \begin{cases} F_{l_1} & k=1 \\ \{0\} & 2 \leq k < n-1 \end{cases}$$

$$\pi_{n-1}(M) \cong \pi_{n-1}(W_f) \cong \begin{cases} \mathbb{Z} & l_2 = 0 \\ \{0\} & l_2 \neq 0 \end{cases}$$

(2) *When $n \geq 3$, $n < m \leq 2n-1$ and $m-n \geq 2$, f is topologically quasi-trivial and we have the following.*

$$\pi_k(M) \cong \pi_k(W_f) \cong \begin{cases} F_{l_1} & k=1 \\ \{0\} & 2 \leq k \leq m-n-1 \end{cases}$$

(3) *When $m \geq 4$ and $n=2$, we have the followings.*
   (a) *If f is topologically quasi-trivial and $l_2 = 0$, then we have the following.*

$$\pi_k(M) \cong \pi_k(W_f) \cong \begin{cases} \mathbb{Z} \times F_{l_1} & k=1 \\ \{0\} & 2 \leq k \leq m-3 \end{cases}$$

   (b) *If f is topologically quasi-trivial and $l_2 \neq 0$, then we have the following.*

$$\pi_1(M) \cong \pi_1(W_f) \cong F_{l_1}$$

We introduce a result of [11], which follows by Proposition 3.

Proposition 6 ([11])**.** *Let M be a closed and connected C^∞ manifold of dimension m and $f : M \to \mathbb{R}^n$ $(n \geq 2)$ be a round fold map. Let $m - n \geq 2$. Suppose that $q_f(S(f))$ consists of two connected components and that one of the connected components is a k S-locus.*
We also assume that f has a family of S-identifications.
Then there exist a compact PL *manifold W of dimension $m+1$ such that $\partial W = M$, a polyhedron V and continuous maps $r : W \to V$ and $s : V \to W_f$ and the followings hold.*

(1) *V is a polyhedron of dimension $n+k$.*
(2) *There exist a triangulation of W, a triangulation of V and a triangulation of W_f such that r is a simplicial map and that s is a simplicial map and the followings hold.*
   (a) *For each $p \in V$, $r^{-1}(p)$ collapses to a point and r is a homotopy equivalence.*
   (b) *If p is in the closure of the AS-region in W_f, then $s^{-1}(p)$ is a point. If p is in the AS-region in W_f and $q \in s^{-1}(p)$, then $r^{-1}(q)$ is* PL *homeomorphic to D^{m-n+1}.*
   (c) *If p is in the k S-region in W_f, then $s^{-1}(p)$ is* PL *homeomorphic to S^k. If p is in the k S-region in W_f and $q \in s^{-1}(p)$, then $r^{-1}(q)$ is* PL *homeomorphic to $D^{m-n-k+1}$.*
(3) *W collapses to a subpolyhedron V' such that $r|_{V'} : V' \to V$ is a* PL *homeomorphism.*
(4) *$r_* \circ i_* : \pi_j(M) \to \pi_j(V)$ gives an isomorphism for $0 \leq j \leq m - \dim V - 1 = m-n-k-1$ where $i : M \to W$ is the inclusion.*

(5) *V is* PL *homeomorphic to* $S^{n+k}\bigcup_{\psi}(S^{n-1}\times[0,1])$ *for a* PL *homeomorphism* $\psi : B \to A$ (*a polyhedron obtained by glueing* S^{n+k} *and* $S^{n-1}\times[0,1]$ *by* ψ) *where A is* PL *homeomorphic to* S^{n-1} *and trivially embedded in* S^{n+k} *in the* PL *category and where* $B := S^{n-1}\times\{0\}\subset S^{n-1}\times[0,1]$.

4. Round fold maps on spheres and bundles over spheres

Proposition 5 and Proposition 6 state that we can know some homotopy groups of manifolds admitting round fold maps with regular fibers homeomorphic to spheres. Now how about the homeomorphism or diffeomorphism types of the manifolds? In this section, we introduce some answers for this question.

Example 2. Let M be a closed C^{∞} manifold of dimension m. A round fold map $f : M \to \mathbb{R}^n$ ($n \geq 2$, $n \neq 4$) whose singular set is connected exists if and only if M is a homotopy sphere admitting a special generic map into $\mathbb{R}^n$ whose Reeb space is homeomorphic to D^n.

Note also that this round fold map is topologically, PL and C^{∞} trivial.
In [9], the following proposition has been shown.

Proposition 7 ([10])**.** *Let M be a closed and connected C^{∞} manifold of dimension m. Suppose that there exists a round fold map $f : M \to \mathbb{R}^n$ ($n \geq 3$) such that the followings hold.*

(1) *The indices of all the fold points are* 0 *or* 1.
(2) *Regular fibers are disjoint unions of almost-spheres.*
(3) $\pi_1(W_f)\cong\{0\}$.
(4) *The fiber of a point in a proper core is non-empty and connected.*

We also assume that $m-n\geq 2$. Then M is a homotopy sphere.

First we prove the following.

Theorem 1. *Let M be a closed C^{∞} manifold of dimension m. Let $n \in \mathbb{N}$ and $m \geq n \geq 2$.*

(1) *Let M have the structure of a C^{∞} bundle over S^n whose fiber is a closed C^{∞} manifold $F(\neq \emptyset)$. Then there exists a C^{∞} trivial round fold map $f : M \to \mathbb{R}^n$ such that the fiber of a point in a proper core of f is C^{∞} diffeomorphic to a disjoint union of two copies of F and that $f^{-1}(L)$ is C^{∞} diffeomorphic to $F\times[0,1]$ for an axis L of f.*
(2) *Suppose that a topologically* ($\mathcal{C}$) *trivial round fold map $f : M \to \mathbb{R}^n$ exists and that for an axis L of f and a closed C^{∞} manifold F of dimension $m-n$, $f^{-1}(L)$ is C^{∞} diffeomorphic to $F\times[0,1]$. Then M has the structure of a* (*resp.* $\mathcal{C}$) *F-bundle over S^n.*

Proof. We prove the first part.
We may assume that $S^n = (D^n \sqcup D^n)\bigcup(S^{n-1}\times[0,1])$ where we identify $\partial(D^n\sqcup D^n) = S^{n-1}\sqcup S^{n-1}$ and $\partial(S^{n-1}\times[0,1]) = S^{n-1}\sqcup S^{n-1}$. We may assume that for a C^{∞} diffeomorphism Φ from $S^{n-1}\times(F\sqcup F)$ onto $\partial D^n\times(F\sqcup F)$ which is a bundle isomorphism between the trivial C^{∞} F-bundles over $\partial(D^n\sqcup D^n) = S^{n-1}\sqcup S^{n-1}$ inducing the identification between the base spaces, $M = ((D^n\sqcup D^n)\times F)\bigcup_{\Phi}(S^{n-1}\times[0,1]\times F) = (D^n\times(F\sqcup F))\bigcup_{\Phi}(S^{n-1}\times[0,1]\times F)$ where $F\sqcup F$ is a disjoint union of two copies of F.

There exists a good Morse function $\tilde{f} : F \times [0,1] \to [a,+\infty)$ where $a \in \mathbb{R}$ is the minimal value. We consider a map $\tilde{f} \times \mathrm{id}_{S^{n-1}}$ and the canonical projection $p : D^n \times (F \sqcup F) \to D^n$. For Φ, $\tilde{f} \times \mathrm{id}_{S^{n-1}}$, p and a C^∞ diffeomorphism $\phi : \partial(\mathbb{R}^n - \mathrm{Int} D^n) \to \partial D^n$, we may assume that the following diagram commutes.

$$\begin{array}{ccc} F \times (\{0\} \sqcup \{1\}) \times \partial(\mathbb{R}^n - \mathrm{Int} D^n) & \xrightarrow{\Phi} & F \times (\{0\} \sqcup \{1\}) \times \partial D^n \\ \Big\downarrow \tilde{f}|_{F \times (\{0\} \sqcup \{1\})} \times \mathrm{id}_{\partial(\mathbb{R}^n - \mathrm{Int} D^n)} & & \Big\downarrow p|_{F \times (\{0\} \sqcup \{1\}) \times \partial D^n} \\ \{a\} \times \partial(\mathbb{R}^n - \mathrm{Int} D^n) & \xrightarrow{\phi} & \partial D^n \end{array}$$

Then on M we can construct a C^∞ map $f := p \bigcup_{\Phi,\phi} (\tilde{f} \times \mathrm{id}_{\partial(\mathbb{R}^n - \mathrm{Int} D^n)})$. The singular set is a disjoint union of standard spheres and consists of fold points and $f|_{S(f)}$ is a C^∞ embedding. We see f is a round fold map satisfying the given conditions.

We now prove the second part.

Suppose that there exists a topologically ($\mathcal{C}$) trivial round fold map $f : M \to \mathbb{R}^n$ and that for an axis L of f and a C^∞ closed manifold F, $f^{-1}(L)$ is C^∞ diffeomorphic to $F \times [0,1]$. Then for a C^∞ diffeomorphism Φ from $S^{n-1} \times (F \sqcup F)$ onto $\partial D^n \times (F \sqcup F)$ which is a bundle isomorphism between the trivial C^∞ $F \sqcup F$-bundles inducing a C^∞ diffeomorphism between the base spaces, M is regarded as $(D^n \times (F \sqcup F)) \bigcup_{\Phi} (S^{n-1} \times [0,1] \times F)$ in the topology category (resp. category $\mathcal{C}$). M has the structure of a (resp. $\mathcal{C}$) F-bundle over S^n.

This completes the proof of both parts of the theorem. $\square$

Example 3 ([9])**.** Let M be a closed C^∞ manifold of dimension m and let M have the structure of a C^∞ bundle over S^n whose fiber is an almost-sphere Σ of dimension $m-n$ ($m > n \geq 2$). Then there exists a C^∞ trivial round fold map $f : M \to \mathbb{R}^n$ such that the fiber of a point in a proper core of f is C^∞ diffeomorphic to a disjoint union of two copies of Σ and that $S(f)$ consists of 2 connected components and is the disjoint union of $F_0(f)$ (the set of all the fold points of indices 0) and $F_1(f)$ (the set of all the fold points of indices 1).

Remark 1. In the situation of Theorem 1, we may weaken the condition that f is topologically ($\mathcal{C}$) trivial and replace it by the followings.

(1) For a proper core P and an axis L of f, $f^{-1}(\mathbb{R}^n - \mathrm{Int} P)$ has the structure of a topologically trivial (resp. trivial $\mathcal{C}$) bundle over ∂P whose fiber is C^∞ diffeomorphic to $f^{-1}(L)$.
(2) $f|_{f^{-1}(\partial P)} : f^{-1}(\partial P) \to \partial P$ gives the structure of a subbundle of the previous bundle $f^{-1}(\mathbb{R}^n - \mathrm{Int} P)$.

We have the following theorem by virtue of the fact that there exist C^∞ homotopy spheres homeomorphic but not C^∞ diffeomorphic to S^7 and having the structures of linear S^3-bundles over S^4 ([13]) and the previous example.

Theorem 2 ([9])**.** *There exist C^∞ homotopy spheres that are homeomorphic but not C^∞ diffeomorphic to S^7 and they admit C^∞ trivial round fold maps whose singular sets consist of 2 connected components. S^7 also admits such a map.*

Remark 2. If there exists a special generic map from a C^∞ homotopy sphere homeomorphic to S^7 into $\mathbb{R}^4$, then the source manifold is C^∞ diffeomorphic to S^7 (see [17] and see also [7] for $\mathrm{Diff}^+(S^3)$). This means that we cannot reduce

the numbers of connected components of the singular sets of round fold maps in Theorem 2 for spheres not C^∞ diffeomorphic to S^7.

We consider some cases where $n = 2$. We have the following theorems.

Theorem 3. *Let M be a closed C^∞ manifold of dimension $m \geq 7$. Let F be a closed and simply-connected C^∞ manifold of dimension $m-2$. Then the followings are equivalent.*

(1) *M has the structure of a C^∞ F-bundle over S^2.*
(2) *M admits a round fold map $f : M \to \mathbb{R}^2$ satisfying the followings.*
   (a) *The regular fiber of a point in a proper core of f is C^∞ diffeomorphic to a disjoint union of two copies of F.*
   (b) *For an axis L of f, $f^{-1}(L)$ is C^∞ diffeomorphic to $F \times [0,1]$.*

Proof. If M admits a round fold map $f : M \to \mathbb{R}^2$, the regular fiber of a point in a proper core of f is C^∞ diffeomorphic to a disjoint union of two copies of F and for an axis L of f, $f^{-1}(L)$ is C^∞ diffeomorphic to $F \times [0,1]$, then since for a proper core P of f, $f|_{f^{-1}(P)} : f^{-1}(P) \to P$ gives the structure of a trivial C^∞ bundle, f is C^∞ trivial by virtue of the pseudoisotopy theorem [3].
By Theorem 1, this completes the proof. □

Theorem 4. *Let M be a closed C^∞ manifold of dimension $m = 3$ or $m = 4$. Then the followings are equivalent.*

(1) *M has the structure of a C^∞ S^{m-2}-bundle over S^2.*
(2) *M admits a round fold map $f : M \to \mathbb{R}^2$ satisfying the followings.*
   (a) *The regular fiber of a point in a proper core of f is C^∞ diffeomorphic to a disjoint union of two copies of S^{m-2}.*
   (b) *For an axis L of f, $f^{-1}(L)$ is C^∞ diffeomorphic to $S^{m-2} \times [0,1]$.*

Proof. If M admits a round fold map $f : M \to \mathbb{R}^2$, the regular fiber of a point in a proper core of f is C^∞ diffeomorphic to a disjoint union of two copies of S^{m-2} and for an axis L of f, $f^{-1}(L)$ is C^∞ diffeomorphic to $S^{m-2} \times [0,1]$, then since for a proper core P of f, $f|_{f^{-1}(P)} : f^{-1}(P) \to P$ gives the structure of a trivial C^∞ bundle, it follows that f is C^∞ trivial by well-known facts on C^∞ $(S^{m-2} \times [0,1])$-bundles over S^1.
By Theorem 1, this completes the proof. □

Theorem 5. *Let M be a closed C^∞ manifold of dimension $m \in \{5,6\}$ and assume that M admits a round fold map $f : M \to \mathbb{R}^2$ satisfying the followings.*

(1) *The regular fiber of a point in a proper core of f is C^∞ diffeomorphic to a disjoint union of two copies of S^{m-2}.*
(2) *For an axis L of f, $f^{-1}(L)$ is C^∞ diffeomorphic to $S^{m-2} \times [0,1]$.*

Then M is PL *homeomorphic to a* PL *S^{m-2}-bundle over S^2.*

Proof. It follows that f is PL trivial. By 2 of Theorem 1, it follows that M is PL homeomorphic to a PL S^{m-2}-bundle over S^2. □

5. Manifolds admitting round fold maps with additional conditions

As in the previous section, we can sometimes know the homeomorphism or diffeomorphism types of manifolds admitting round fold maps. In general, it seems to be difficult to know the homeomorphism and diffeomorphism types of manifolds

strictly. Here, we consider such problems and give some answers. In this section, we only consider round fold maps from manifolds of dimensions m into $\mathbb{R}^n$ where $n \geq 2$ and $m \geq 2n$.

5.1. **Round fold maps on connected sums of two connected oriented manifolds manifolds admitting round fold maps.** In this subsection, we construct round fold maps on the connected sum of two oriented manifolds admitting round fold maps under additional conditions.

Proposition 8. *Let M_1 and M_2 be closed and connected C^{∞} oriented manifolds of dimensions m. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ ($n \geq 2$, $m \geq 2n$) such that the fiber of a point in a proper core of f_1 has a connected component C^{∞} diffeomorphic to S^{m-n} and a round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that for the boundary C of the unbouded connected component of $\mathbb{R}^n - \mathrm{Int} f_2(M_2)$, the inclusion of ${f_2}^{-1}(C)$ into M_2 is null-homotopic.*
Then, for the connected sum M of M_1 and M_2, there exists a round fold map $f : M \to \mathbb{R}^n$.

Proof. Let P_1 be a proper core of f_1 and P_2 be a small C^{∞} closed tubular neighborhood of the connected component C of $f_2(S(f_2))$ (see also FIGURE 2). Let V_1 be a connected component of ${f_1}^{-1}(P_1)$ such that $f|_{V_1} : V_1 \to P_1$ gives the structure of a trivial C^{∞} S^{m-n}-bundle over D^n and $V_2 := {f_2}^{-1}(P_2)$. V_2 is a C^{∞} closed tubular neighborhood of ${f_2}^{-1}(C) \subset M_2$ and V_2 has the structure of a trivial linear D^{m-n+1}-bundle since $m \geq 2n = 2(n-1)+2$ is assumed and the inclusion ${f_2}^{-1}(C)$ into M_2 is assumed to be null-homotopic. More precisely, $f_2|_{\partial V_2}$ gives the structure of a subbundle of the bundle.
Since $m \geq 2n = 2(n-1)+2$ is assumed, we have the following for a C^{∞} diffeomorphism $\Phi : \partial V_2 \to \partial V_1$ regarded as a bundle isomorphism between the two trivial C^{∞} S^{m-n+1}-bundles over S^{n-1} inducing a C^{∞} diffeomorphism between the base spaces and a C^{∞} diffeomorphism $\Psi : \partial D^m \to \partial D^m$ extending to a C^{∞} diffeomorphism on D^m (from $M_2 - \mathrm{Int}(M_2 - \mathrm{Int} D^m)$ onto $M_1 - \mathrm{Int}(M_1 - \mathrm{Int} D^m)$) where for two C^{∞} manifolds X_1 and X_2, $X_1 \cong X_2$ means that X_1 and X_2 are C^{∞} diffeomorphic.

$$\begin{aligned}
& (M_1 - \mathrm{Int} V_1) \bigcup_{\Phi} (M_2 - \mathrm{Int} V_2) \\
\cong\ & (M_1 - \mathrm{Int} V_1) \bigcup_{\Phi} ((D^m - \mathrm{Int} V_2) \bigcup (M_2 - \mathrm{Int} D^m)) \\
\cong\ & (M_1 - \mathrm{Int} V_1) \bigcup_{\Phi} ((S^m - (\mathrm{Int} V_2 \sqcup \mathrm{Int} D^m)) \bigcup_{\Psi} (M_2 - \mathrm{Int} D^m)) \\
\cong\ & (M_1 - \mathrm{Int} D^m) \bigcup_{\Psi} (M_2 - \mathrm{Int} D^m)
\end{aligned}$$

This means that the resulting manifold M is the connected sum of M_1 and M_2 and that M admits a round fold map $f : M \to \mathbb{R}^n$. More precisely, f is obtained by glueing two maps $f_1|_{M_1 - \mathrm{Int} V_1}$ and $f_2|_{M_2 - \mathrm{Int} V_2}$. □

Example 4. Let M_1 and M_2 be closed and connected C^{∞} oriented manifolds of dimensions m.
Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ ($n \geq 2$, $m \geq 2n$) such that the fiber of a point in a proper core of f_1 has a connected component C^{∞} diffeomorphic to S^{m-n}.

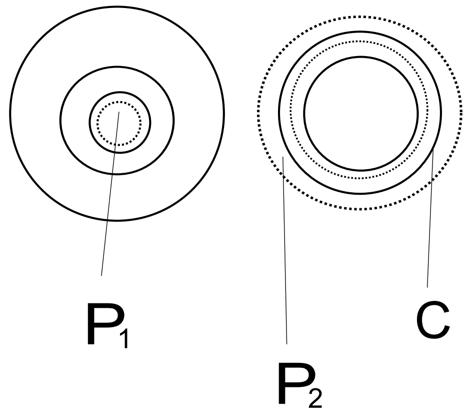

FIGURE 2. $P_1, P_2, C \subset \mathbb{R}^n$ (P_1 is the bounded region bounded by the dotted line in the left figure and P_2 is the region bounded by the disjoint union of the dotted lines in the right figure.)

If $\pi_{n-1}(M_2) \cong \{0\}$, then f_1 and f_2 satisfy the assumption of Proposition 8.
If $f_2(M_2)$ is C^∞ diffeomorphic to D^n, $n \geq 3$ and f_2 satisfies the assumption of Proposition 5, then f_1 and f_2 satisfy the assumption of Proposition 8.

As a result, we have the following theorem.

Theorem 6. *Let M_1 and M_2 be closed and connected C^∞ oriented manifolds of dimensions m. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ ($n \geq 2$, $m \geq 2n$) such that the fiber of a point in a proper core P_1 of f_1 has a connected component C^∞ diffeomorphic to S^{m-n}.*

(1) *Let there exist a round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that $f_2(M_2)$ is C^∞ diffeomorphic D^n. For an axis L of f_2, let $\pi_{n-2}({f_2}^{-1}(L)) \cong \pi_{n-1}({f_2}^{-1}(L)) \cong \{0\}$. We also assume that for any point p in a proper core P_2 of f_2 and any connected component F_p of the fiber ${f_2}^{-1}(p)$, $\pi_{n-1}(F_p) \cong \{0\}$. Then, for the connected sum M of M_1 and M_2, there exists a round fold map $f : M \to \mathbb{R}^n$.*
(2) *Let there exist a round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that $f_2(M_2)$ is C^∞ diffeomorphic D^n. For any point p in a proper core P_2 of f_2 and any connected component F_p of ${f_2}^{-1}(p)$, let $\pi_{n-1}(F_p) \cong \{0\}$. For any connected component C of $f_2(S(f_2))$, let there exist a small C^∞ closed regular neighborhood $N(C)$ such that $\partial N(C)$ consists of two connected components C_1 and C_2 and let ${f_2}^{-1}(N(C))$ have the structure of a C^∞ bundle over C_1 (C_2) such that for any connected component F_C of its fiber, $\pi_{n-2}(F_C) \cong \pi_{n-1}(F_C) \cong \{0\}$ holds and that $f_2|_{{f_2}^{-1}(C_1)} : {f_2}^{-1}(C_1) \to C_1$ (resp. $f_2|_{{f_2}^{-1}(C_2)} : {f_2}^{-1}(C_2) \to C_2$) gives the structure of a subbundle of the bundle ${f_2}^{-1}(N(C))$. We also assume that all the connected components of the subbundles admit sections. Then, for the connected sum M of M_1 and M_2, there exists a round fold map $f : M \to \mathbb{R}^n$.*
(3) *Let there exist a round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that $f_2(M_2)$ is C^∞ diffeomorphic D^n. Let f_2 have a family of S-identifications. Assume that for any integer k satisfying $m-n-k-1 < n-1$ there do not exist k S-loci*

in W_{f_2}. Let us use the notations s as in Proposition 6 and the natural projection π_P as in subsection 3.1 for f_2 and P_2 be a proper core of f_2. We also assume that $\pi_P \circ \bar{f_2} \circ s|_{s^{-1}(\bar{f_2}^{-1}(\pi_P{}^{-1}(\partial P_2)))} : s^{-1}(\bar{f_2}^{-1}(\pi_P{}^{-1}(\partial P_2))) \to \partial P_2$ gives the structure of a topologically trivial bundle. Then, for the connected sum M of M_1 and M_2, there exists a round fold map $f : M \to \mathbb{R}^n$.

Proof. We prove the first part. We consider the homotopy exact sequence $\pi_{n-1}({f_2}^{-1}(L))(\cong \{0\}) \to \pi_{n-1}({f_2}^{-1}(\mathbb{R}^n - \mathrm{Int}P_2)) \to \pi_{n-1}(\partial P_2) \to \pi_{n-2}({f_2}^{-1}(L))(\cong \{0\})$ and this means that $\pi_{n-1}({f_2}^{-1}(\mathbb{R}^n - \mathrm{Int}P_2)) \cong \pi_{n-1}(\partial P_2) \cong \mathbb{Z}$ and that the isomorphism is given by the natural projection. This means that the embedding ${f_2}^{-1}(\partial f_2(M_2)) \subset M_2$ and a section of the trivial bundle $f_2|_{{f_2}^{-1}(\partial P_2)} : {f_2}^{-1}(\partial P_2) \to \partial P_2$ are homotopic. Since for any point p in a proper core P_2 and any connected component F_p of the fiber of p, $\pi_{n-1}(F_p) \cong \{0\}$ (note also that $f_2(M)$ is assumed to be C^∞ diffeomorphic to D^n), the embeddings are null-homotopic in M_2. This completes the first part of the proof by Proposition 8.
We prove the second part. For any connected component C of $f_2(S(f_2))$ and for any connected component E_C of the bundle ${f_2}^{-1}(N(C))$, $\pi_{n-1}(E_C) \cong \pi_{n-1}(C_i) \cong \mathbb{Z}$ holds for $i = 1, 2$ and the isomorphism is given by the natural projection. This means that all the connected components of the bundles $f_2|_{{f_2}^{-1}(C_1)} : {f_2}^{-1}(C_1) \to C_1$ and $f_2|_{{f_2}^{-1}(C_2)} : {f_2}^{-1}(C_2) \to C_2$ admit sections by the assumption and that any pair of the sections in a same connected component of the bundle ${f_2}^{-1}(N(C))$ are homotopic. Since M is connected, the sections are homotopic to sections of the trivial C^∞ bundle over the boundary of the proper core P_2 of f_2 given by $f_2|_{{f_2}^{-1}(\partial P_2)} : {f_2}^{-1}(\partial P_2) \to \partial P_2$. Since for any connected component F_p of the fiber of p in the proper core P_2, $\pi_{n-1}(F_p) \cong \{0\}$ (note also that $f_2(M)$ is assumed to be C^∞ diffeomorphic to D^n), the sections are null-homotopic in M_2. It follows that the inclusion of ${f_2}^{-1}(\partial f_2(M_2))$ into M_2 is also null-homotopic. This completes the second part of the proof by Proposition 8.
We prove the last part. Let us use the notations V and r as in Proposition 3 for f_2. Since by the assumption that for any integer k satisfying $m - n - k - 1 < n - 1$ there do not exist k S-loci in W_{f_2}, $m - \dim V - 1 \geq n - 1$ holds, by Corollary 1, $r|_{M_{2*}} : \pi_{n-1}(M_2) \to \pi_{n-1}(V)$ gives an isomorphism. $\pi_P \circ \bar{f_2} \circ s|_{s^{-1}(\bar{f_2}^{-1}(\pi_P{}^{-1}(\partial P_2)))} : s^{-1}(\bar{f_2}^{-1}(\pi_P{}^{-1}(\partial P_2))) \to \partial P_2$ gives the structure of a topologically trivial bundle by the assumption. It follows that the inclusion of $s^{-1}(\bar{f_2}^{-1}(\partial f_2(M)))$ (note that it is homeomorphic to S^{n-1}) into V is null-homotopic since $f_2(M)$ is assumed to be C^∞ diffeomorphic to D^n and as a result the inclusion of ${f_2}^{-1}(\partial f_2(M_2))$ into M_2 is null-homotopic. This completes the last part of the proof by Proposition 8.
This completes the proof. □

Example 5. Let M_1 and M_2 be closed and connected C^∞ oriented manifolds of dimensions m. Let there exist a round fold map $f_1 : M_1 \to \mathbb{R}^n$ ($n \geq 2$, $m \geq 2n$) such that the fiber of a point in a proper core of f_1 has a connected component C^∞ diffeomorphic to S^{m-n}. Suppose that M_2 admits a round fold map $f_2 : M_2 \to \mathbb{R}^n$ and that $f_2(M_2)$ is C^∞ diffeomorphic D^n.

(1) If f_2 is C^∞ trivial and all the regular fibers of f_2 are closed and $(n-1)$-connected manifolds, then f_1 and f_2 satisfy the assumption of 1 of Theorem 6.

(2) If f_2 is a round fold map satisfying the assumption of Theorem 2 of [11] or Proposition 6 of this paper and $m-n-k-1 \geq n-1$ in the propositions, then f_1 and f_2 satisfy the assumption of 3 of Theorem 6.

We have the following theorem where source manifolds are 5-dimensional.

Theorem 7. *Let M be a closed and connected C^{∞} oriented manifold of dimension* 5*.*
M admits a round fold map $f : M \to \mathbb{R}^2$ satisfying the assumption of Proposition 5 and $l_1 = 0$ in the proposition if and only if M is C^{∞} diffeomorphic to a connected sum of a finite number of C^{∞} oriented S^3-bundles over S^2.

For the proof of Theorem 7, we need the following proposition.

Proposition 9 (Barden, [1])**.** *Let M be a closed and simply-connected C^{∞} oriented manifold of dimension* 5*. Then $H_2(M;\mathbb{Z})$ is torsion-free if and only if M is a connected sum of a finite number of C^{∞} oriented S^3-bundles over S^2.*

Proof of Theorem 7. By 1 of Theorem 1 or Example 3 and by the proof of Proposition 8, a connected sum of a finite number of C^{∞} oriented S^3-bundles over S^2 admits a round fold map into $\mathbb{R}^2$ satisfying the assumption of Proposition 5 and $l_1 = 0$ in the proposition.
For a round fold map f from a closed and simply-connected manifold M of dimension 5 into $\mathbb{R}^2$ satisfying the assumption of Proposition 5 and $l_1 = 0$ in the proposition, the Reeb space W_f is simple homotopy equivalent to $A \bigcup_{\psi} B$ where A is a disjoint union of finite copies of D^n, where $B := S^{n-1} \times L$, where L is a compact and connected graph with no loops, where $\psi : S^{n-1} \times \Lambda \to \partial A$ is a PL homeomorphism and where Λ is a set consisting of a finite number of degree 1 vertices of the graph L and as a result it is simple homotopy equivalent to a bouquet of finite copies of S^2 (for this argument, see also the proof of Proposition 5 or Theorem 3 of [10] and section 3 of [9]). $\pi_1(M) \cong \pi_1(W_f) \cong H_1(W_f) \cong \{0\}$ holds and $\pi_2(M) \cong \pi_2(W_f) \cong H_2(W_f)$ is torsion-free.
Hence a closed and simply-connected manifold M of dimension 5 such that $H_2(M,\mathbb{Z})$ is not torsion-free does not admit a round fold map into $\mathbb{R}^2$ satisfying the assumption of Proposition 5 and $l_1 = 0$ in the proposition.
By Proposition 9, $H_2(M,\mathbb{Z})$ is not torsion-free if M is not a connected sum of a finite number of C^{∞} oriented S^3-bundles over S^2. This completes the proof. □

5.2. **Decompositions of a manifold admitting a round fold map into a connected sum of two manifolds admitting round fold maps.** Conversely, in this subsection, we decompose a manifold admitting a round fold map into a connected sum of two manifolds admitting round fold maps under additional conditions. For this subsection, see also [12] and compare the arguments with arguments in the paper on surgery operations on stable maps from closed and simply-connected C^{∞} manifolds into the plane which do not change the diffeomorophism types of the source manifolds (*R-operations*).

Proposition 10. *Let M be a closed and connected C^{∞} oriented manifold of dimension m. Let $f : M \to \mathbb{R}^n$ be a round fold map ($n \geq 2$, $m \geq 2n$).*
Let C be a C^{∞} submanifold of $\mathbb{R}^n - f(S(f))$. Suppose that C is C^{∞} diffeomorphic to S^{n-1} and that C is a deformation retract of the closure of a connected component of $\mathbb{R}^n - f(S(f))$ C^{∞} diffeomorphic to $S^{n-1} \times (0,1)$ or is in a proper core of f.

For a small C^∞ closed tubular neighborhood $N(C)$, we denote the bounded domain of $\mathbb{R}^n - \mathrm{Int}N(C)$ by R and let there exist a connected component $\bar{M}$ of $f^{-1}(R)$ satisfying the follwoings.

1. *$f|_{\partial \bar{M}} : \partial \bar{M} \to C$ gives the structure of a trivial C^∞ bundle over C and the fiber is a standard sphere.*
2. *If we glue $\bar{M}$ and $V := S^{n-1} \times D^{m-n+1}$ on the boundaries by a bundle isomorphism Φ between the two trivial C^∞ bundles $\partial \bar{M}$ and $V := S^{n-1} \times \partial D^{m-n+1}$ over the standard spheres of dimensions $n-1$ inducing a C^∞ diffeomorphism between the base spaces, then the natural inclusion $S^{n-1} \times \{p\} \subset V = S^{n-1} \times D^{m-n+1} \subset \bar{M} \bigcup_{\Phi} V$ is null-homotopic.*

Then M is C^∞ diffeomorphic to the connected sum of two connected C^∞ manifolds M_1 and $M_2 := \bar{M} \bigcup_{\Phi} V$, M_i admits a round fold map $f_i : M_i \to \mathbb{R}^n$ $(i = 1, 2)$ and the followings hold.

1. *$M_1 \supset M - \bar{M}$ and $M_1 - (M - \bar{M})$ is C^∞ diffeomorphic to $D^n \times S^{m-n}$.*
2. *$S(f_1) = S(f) \bigcap (M - \bar{M})$ and $f|_{M - \mathrm{Int}\bar{M}} = f_1|_{M - \mathrm{Int}\bar{M}}$.*
3. *For the connected component C' of $\partial f_2(M_2)$ bounding the unbounded connected component of $\mathbb{R}^n - \mathrm{Int} f_2(M_2)$, $S(f_2) \bigcap (M_2 - \bar{M}) = {f_2}^{-1}(C')$ holds and $f_2|_{\bar{M}} = f|_{\bar{M}}$ holds.*

Proof. Let M_1 be a closed C^∞ manifold given by glueing $M - \mathrm{Int}\bar{M}$ and $V' := D^n \times S^{m-n}$ by some C^∞ diffeomorphism $\Phi' : \partial D^n \times S^{m-n} \to \partial \bar{M}$ regarded as a bundle isomorphism between the two trivial C^∞ S^{m-n}-bundles inducing a C^∞ diffeomorphism between the base spaces.
Since $m \geq 2n = 2(n-1)+2$ is assumed and the natural inclusion $S^{n-1} \times \{p\} \subset V = S^{n-1} \times D^{m-n+1} \subset \bar{M} \bigcup_{\Phi} V$ is assumed to be null-homotopic, we may assume that the following holds for a C^∞ diffeomorphism $\Psi : \partial D^m \to \partial D^m$ extending to a C^∞ diffeomorphism on D^m (from $M_2 - \mathrm{Int}(M_2 - \mathrm{Int}D^m)$ onto $M_1 - \mathrm{Int}(M_1 - \mathrm{Int}D^m)$) where for two C^∞ manifolds X_1 and X_2, $X_1 \cong X_2$ means that X_1 and X_2 are C^∞ diffeomorphic.

$$
\begin{aligned}
& \bar{M} \bigcup (M - \mathrm{Int}\bar{M}) \\
\cong\ & (M_1 - \mathrm{Int}V') \bigcup (M_2 - \mathrm{Int}V) \\
\cong\ & (M_1 - \mathrm{Int}V') \bigcup ((D^m - \mathrm{Int}V) \bigcup (M_2 - \mathrm{Int}D^m)) \\
\cong\ & (M_1 - \mathrm{Int}V') \bigcup ((S^m - (\mathrm{Int}V \sqcup \mathrm{Int}D^m)) \bigcup_{\Psi} (M_2 - \mathrm{Int}D^m)) \\
\cong\ & (M_1 - \mathrm{Int}D^m) \bigcup_{\Psi} (M_2 - \mathrm{Int}D^m)
\end{aligned}
$$

This means that M is C^∞ diffeomorphic to the connected sum of M_1 and M_2 and that M_1 and M_2 admit round fold maps satisfying the mentioned conditions. This completes the proof. □

As results, we have the following theorem.

Theorem 8. *Let M be a closed and connected C^∞ oriented manifold of dimension m. Let $f : M \to \mathbb{R}^n$ be a round fold map $(n \geq 2,\ m \geq 2n)$.*
Let C be a C^∞ submanifold of $\mathbb{R}^n - f(S(f))$. Suppose that C is C^∞ diffeomorphic to S^{n-1} and that C is a deformation retract of the closure of a connected component of $\mathbb{R}^n - f(S(f))$ C^∞ diffeomorphic to $S^{n-1} \times (0,1)$ or is in a proper core of f.

For a small C^∞ closed tubular neighborhood $N(C)$, we denote the bounded domain of $\mathbb{R}^n - \mathrm{Int}N(C)$ by R and let there exist a connected component $\bar{M}$ of $f^{-1}(R)$ satisfying the follwoings.

(1) *$f|_{\partial \bar{M}} : \partial \bar{M} \to C$ gives the structure of a trivial C^∞ bundle over C and the fiber is a standard sphere.*

(2) *If we glue $\bar{M}$ and $S^{n-1} \times D^{m-n+1}$ on the boundaries by a bundle isomorphism Φ between the two trivial C^∞ bundles $\partial \bar{M}$ and $S^{n-1} \times \partial D^{m-n+1}$ over the standard spheres of dimensions $n-1$ inducing a C^∞ diffeomorphism between the base spaces, then the resulting manifold $M_2 := \bar{M} \bigcup_{\Phi} (S^{n-1} \times D^{m-n+1})$ admits a round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that the followings hold.*
   (a) *$f_2(M_2)$ is C^∞ diffeomorphic to D^n.*
   (b) *$S(f_2) \bigcap (M_2 - \bar{M}) = {f_2}^{-1}(\partial f_2(M_2))$.*

   We also assume that one or two or three of the followings hold.
   (a) *For an axis L of f_2, $\pi_{n-2}({f_2}^{-1}(L)) \cong \pi_{n-1}({f_2}^{-1}(L)) \cong \{0\}$ and for any connected component F of the fiber of a point in a proper core, $\pi_{n-1}(F) \cong \{0\}$.*
   (b) (i) *For any point p in a proper core of f_2 and any connected component F_p of ${f_2}^{-1}(p)$, $\pi_{n-1}(F_p) \cong \{0\}$.*
   (ii) *For any connected component C' of $f_2(S(f_2))$, there exists a small C^∞ closed tubular neighborhood $N(C')$ of C' such that $\partial N(C')$ consists of two connected components C_1 and C_2 and ${f_2}^{-1}(N(C'))$ has the structure of a C^∞ bundle over C_1 (C_2) such that for any connected component $F_{C'}$ of its fiber, $\pi_{n-2}(F_{C'}) \cong \pi_{n-1}(F_{C'}) \cong \{0\}$ holds and that $f_2|_{{f_2}^{-1}(C_1)} : {f_2}^{-1}(C_1) \to C_1$ (resp. $f_2|_{{f_2}^{-1}(C_2)} : {f_2}^{-1}(C_2) \to C_2$) gives the structure of a subbundle of the bundle ${f_2}^{-1}(N(C'))$. Furthermore, all the connected components of the subbundles admit sections.*
   (c) (i) *f_2 has a family of S-identifications.*
   (ii) *Let us use the notations s as in Proposition 6 and the natural projection π_P as in subsection 3.1 for f_2 and P be a proper core of f_2. We also assume that $\pi_P \circ \bar{f_2} \circ s|_{s^{-1}(\bar{f_2}^{-1}({\pi_P}^{-1}(\partial P_2)))} : s^{-1}(\bar{f_2}^{-1}({\pi_P}^{-1}(\partial P_2))) \to \partial P_2$ gives the structure of a topologically trivial bundle. Furthermore, for any integer k satisfying $m-n-k-1 < n-1$, there do not exist k S-loci in the Reeb space W_{f_2} of f_2.*

Then M is C^∞ diffeomorphic to the connected sum of two connected C^∞ manifolds M_1 and M_2 and the followings holds.

(1) *$M_1 \supset M - \bar{M}$ and $M_1 - (M - \bar{M})$ is C^∞ diffeomorphic to $D^n \times S^{m-n}$.*

(2) *M_1 admits a round fold map $f_1 : M_1 \to \mathbb{R}^n$ such that $S(f_1) = S(f) \bigcap (M - \bar{M})$ and $f|_{M - \mathrm{Int}\bar{M}} = f_1|_{M - \mathrm{Int}\bar{M}}$ hold.*

Proof. By the proof of Theorem 6, we can apply Proposition 10 to prove Theorem 8. In fact, each of the last three conditions of 2 of this theorem corresponds to each of the three statements of Theorem 6 and we can decompose the given manifold admitting a round fold map into two manifolds admitting round fold maps as in the proof of Proposition 10.
This completes the proof. □

5.3. **Applications.** In this subsection, by applying the previous results and their proofs, we determine the diffeomorphism types of manifolds admitting round fold maps under appropriate conditions.

Theorem 9. *Let M be a closed and connected C^{∞} oriented manifold of dimension m. Suppose that there exists a round fold map $f : M \to \mathbb{R}^n$ $(n \geq 2)$. Suppose that $m \geq 2n$.*
Assume that for any connected component R of $\mathbb{R}^n - \mathrm{Int}N(f(S(f)))$ where $N((S(f)))$ is a small C^{∞} closed tubular neighborhood of $f(S(f))$, $f|_{f^{-1}(R)} : f^{-1}(R) \to R$ gives the structure of a trivial C^{∞} bundle whose fiber is a disjoint union of standard spheres. We also assume that the number of connected components of the fiber of a point in a proper core of f equals the number of connected components of $S(f)$.
Then M is a connected sum of closed and connected C^{∞} oriented manifolds admitting round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of disjoint unions of two standard spheres.
Conversely, such a manifold admits a round fold map into $\mathbb{R}^n$ satisfying the assumption.

Proof. By the last statement of Proposition 2, f satisfies the assumption of Proposition 5 and $l_1 = 0$ in the proposition by the assumption that the number of connected components of the fiber of a point in a proper core of f equals the number of connected components of $S(f)$. It also follows that f is topologically quasi-trivial. Furthermore, the Reeb space W_f is represented as $A \bigcup_{\psi} B$ where A is a disjoint union of finite copies of D^n, where $B := S^{n-1} \times L$, where L is a compact and connected graph with no loops, where $\psi : S^{n-1} \times \Lambda \to \partial A$ is a PL homeomorphism and where Λ is a set consisting of a finite number of degree 1 vertices of the graph L and as a result it is a bouquet of finite copies of S^n (see also the proof of Theorem 7 of this paper). By the proof of Proposition 10 or Theorem 8 and by topological properties of the Reeb space W_f, if the singular set of f consists of more than 2 connected components, then we obtain 2 or more round fold maps so that the number of the singular set of each map is smaller than that of the original map f, that each map satisfies the assumption in this theorem and that the connected sum of the resulting oriented source manifolds is M. It also follows that M is a connected sum of closed and connected C^{∞} oriented manifolds admitting round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of disjoint unions of two standard spheres.
Conversely, if M is a connected sum of closed and connected C^{∞} oriented manifolds admitting round fold maps with singular sets consisting of 2 connected components and with fibers of points in proper cores consisting of disjoint unions of two standard spheres, then there exists a round fold map $f : M \to \mathbb{R}^n$ satisfying the assumption by the proof of Proposition 8 or 1 of Theorem 6. □

Let Θ_{k_1} be the h-cobordism group of C^{∞} oriented homotopy spheres of dimensions $k_1 \geq 2$. It follows easily that the set of all the classes of Θ_{k_1} consisting of spheres admitting round fold maps with connected singular sets into $\mathbb{R}^{k_2}$ $(k_1 \geq k_2 \geq 2)$ is a subgroup of Θ_{k_1}. In fact we only consider the connected sum of given two round fold maps with connected singular sets (for the connected sum of given two special generic maps into Euclidean spaces, see section 5 of [16] for example).

Definition 11. We denote the subgroup by $\Theta_{(k_1,k_2)} \subset \Theta_{k_1}$ and call it the (k_1, k_2) *round special generic group.*

We have the following theorem.

Theorem 10. *Let M be a closed and connected C^{∞} oriented manifold of dimension m. Let $n \in \mathbb{N}$, $n \geq 2$ and $m \geq 2n$. Then the followings are equivalent.*

1. *A round fold map $f : M \to \mathbb{R}^n$ satisfying the followings exists.*
   1. (a) *$S(f)$ consists of 2 connected components.*
   2. (b) *For the connected component $C \neq f(F_0(f))$ of $f(S(f))$ and a small C^{∞} closed tubular neighborhood $N(C)$ of C such that $\partial N(C)$ is the disjoint union of two connected components C_1 and C_2, $f^{-1}(N(C))$ has the structure of a trivial C^{∞} bundle over C_1 (C_2) with fibers* PL *homeomorphic to $S^{m-n+1} - \mathrm{Int}(D^{m-n+1} \sqcup D^{m-n+1} \sqcup D^{m-n+1})$ and $f|_{f^{-1}(C_1)} : f^{-1}(C_1) \to C_1$ (resp. $f|_{f^{-1}(C_2)} : f^{-1}(C_2) \to C_2$) gives the structure of a subbundle of the bundle $f^{-1}(N(C))$.*
2. *M is a connected sum of an oriented manifold in a class of $\Theta_{(m,n)}$ and a C^{∞} oriented bundle over S^n with fibers C^{∞} diffeomorphic to an almost-sphere.*

Proof. Assume that a round fold map $f : M \to \mathbb{R}^n$ satisfying the condition 1 exists. Note that f satisfies the assumption of Proposition 5. By the proof of Proposition 10 or Theorem 8, we can represent M as a connected sum of two closed and connected C^{∞} manifolds M_1 and M_2 and the followings hold.

1. M_1 admits a round fold map $f_1 : M_1 \to \mathbb{R}^n$ such that $S(f_1)$ is connected.
2. M_2 admits a C^{∞} trivial round fold map $f_2 : M_2 \to \mathbb{R}^n$ such that $S(f_2)$ consists of two connected components and that the fiber of a point in a proper core of f_2 is C^{∞} diffeomorphic to a disjoint union of two copies of S^{m-n}. f_2 satisfies the assumption of Proposition 5.

Then M is a connected sum of an oriented manifold in a class of $\Theta_{(m,n)}$ and a C^{∞} oriented bundle over S^n with fibers C^{∞} diffeomorphic to an almost-sphere by 2 of Theorem 1.
Conversely, if M is such a manifold, then by 1 of Theorem 1 or Example 3 and by the proof of Proposition 8 or 1 of Theorem 6, M admits a round fold map $f : M \to \mathbb{R}^n$ satisfying the condition 1.
This completes the proof. $\square$

Now we have the following theorem.

Theorem 11. *Let M be a closed and connected C^{∞} oriented manifold of dimension m. Let $n \in \mathbb{N}$, $n \geq 2$ and $m \geq 2n$. Then the followings are equivalent.*

1. *A round fold map $f : M \to \mathbb{R}^n$ satisfying the followings exist.*
   1. (a) *Regular fibers of f are disjoint unions of standard spheres and the number of connected components of the fiber of a point in a proper core of f equals the number of connected components of $S(f)$.*
   2. (b) *For any connected component C of $f(F_1(f))$ and a small C^{∞} closed tubular neighborhood $N(C)$ of C such that $\partial N(C)$ is the disjoint union of two connected components C_1 and C_2, $f^{-1}(N(C))$ has the structure of a trivial C^{∞} bundle over C_1 (C_2) and $f|_{f^{-1}(C_1)} : f^{-1}(C_1) \to C_1$ (resp. $f|_{f^{-1}(C_2)} : f^{-1}(C_2) \to C_2$) gives the structure of a subbundle of the bundle $f^{-1}(N(C))$.*

(2) *M is a connected sum of a finite number of C^∞ oriented S^{m-n}-bundles over S^n and an oriented manifold in a class of $\Theta_{(m,n)}$.*

Proof. Assume that a round fold map $f : M \to \mathbb{R}^n$ satisfying the condition 1 exists. By the proof of Theorem 9 and by Theorem 10, M is a connected sum of a finite number of C^∞ closed manifolds admitting round fold maps with singular sets consisting of 2 connected components and each manifold is a connected sum of a C^∞ oriented S^{m-n}-bundle over S^n and an oriented manifold in a class of $\Theta_{(m,n)}$.
Conversely, by 1 of Theorem 1 or Example 3, each C^∞ oriented S^{m-n}-bundle over S^n admits a C^∞ trivial round fold map with the singular set consisting of 2 connected components and the fiber of a point in its proper core is a disjoint union of two standard spheres. By the proof of Proposition 8 or 1 of Theorem 6, a connected sum of a finite number of C^∞ oriented S^{m-n}-bundles over S^n and an oriented manifold in a class of $\Theta_{(m,n)}$ admits a round fold map $f : M \to \mathbb{R}^n$ satisfying the condition 1.
This completes the proof. $\square$

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